Linearization and Liapunov Stability Analysis of a Class of Dynamical Differential Equations

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For a nongyroscopic, holonomic, conservative, dynamical system of differential equations for which the Lagrangian is free of explicit time dependence and the dynamic potential energy P is neither positive nondefinite nor zero, it is demonstrated that for Liapunov stability of the null solution it is both necessary and sufficient that P be positive definite, and furthermore that complete stability criteria can be obtained from linearized variational equations. These results are illustrated by comparison with a recently published stability analysis of a satellite attitude sensing device.

Introduction

BEFORE the turn of the century, Poincare[1] and Liapunov[2] established the foundations of stability theory of ordinary differential equations, motivated largely by a common interest in problems of mechanics.[3,4] In recent years, however, work in stability theory has been dominated by differential equationists and by those interested in problems of automatic control, so that modern emphasis has concentrated on the development of theorems applicable to more general differential equations than those found in mechanics. Yet some of the basic questions which remain in the general theory have simple answers within the framework of the equations of motion of restricted dynamical systems.

In the general case, linearized constant coefficient variational equations provide a rigorous basis for the determination of Liapunov stability only when either asymptotic stability or instability is indicated,[5] and in the general case a Liapunov function provides only sufficient conditions for stability.[5] For a special class of mechanical systems it will be shown that linearized variational equations of motion provide complete and rigorous Liapunov stability criteria, and furthermore that the use of the Hamiltonian as a Liapunov function also leads to complete stability criteria. It is the purpose of this paper to delineate a class of mechanical systems for which these convenient simplifications of the general theory apply, and to illustrate these results with an application to a mechanical device recently proposed as a satellite attitude sensor.

It should be noted that the mechanical systems in question are conservative, and hence undamped.† Pringle[6] has already established for completely damped, holonomic, mechanical systems a counterpart to the results of the present paper. Moreover, the theorems which follow establish stability criteria for mechanical systems in motion relative to inertial space; stability criteria for static equilibrium have been established by Lagrange, Liapunov, and Chetayev.[7]

Stability Criteria

Special procedures for stability analysis can be used for assessing the stability of a null solution of a system of differential equations of the class represented by

$$d/dt(\partial L/\partial \dot{q}_i) - \partial L/\partial q_i = 0 \quad i = 1, \ldots, n \tag{1}$$

when subjected to the restrictions $\partial L/\partial t = 0$ and

$$d/dt(\partial T_1/\partial \dot{q}_i) - \partial T_1/\partial q_i = 0 \quad i = 1, \ldots, n \tag{2}$$

where the Lagrangian L is the kinetic energy T less the potential energy V, and T_1 is that portion of T which is linear in the variational coordinate generalized velocities $\dot{q}_1, \ldots, \dot{q}_n$. The class of equations defined by Eq. (1) is called a holonomic conservative, dynamical system of differential equations, and with the added condition of Eq. (2) the equations became nongyroscopic.

It is always possible to represent the kinetic energy T as the sum

$$T = T_2 + T_1 + T_0 \tag{3}$$

where T_i is the sum of terms of ith degree in $\dot{q}_1, \ldots, \dot{q}_n$, for $i = 0, 1, 2$. In terms of the symbols defined by Eq. (3) and the dynamic potential energy P defined by

$$P \triangleq V - T_0 \tag{4}$$

the Lagrangian L is given by

$$L = T_2 + T_1 - P \tag{5}$$

and the Hamiltonian H is defined as

$$H \triangleq \sum_{i=1}^{n} (\partial L/\partial \dot{q}_i)\dot{q}_i - L = T_2 + P \tag{6}$$

Under the restriction $\partial L/\partial t = 0$, the Hamiltonian is a constant of the motion, a first integral of Eq. (1), as may be confirmed by the differentiation

$$\dot{H} = \sum_{i=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i - \frac{\partial L}{\partial q_i}\dot{q}_i - \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right] - \frac{\partial L}{\partial t} = 0 \tag{7}$$

For nongyroscopic, holonomic, conservative, dynamical systems of differential equations with $\partial L/\partial t = 0$, stability analysis based on linearized variational equations furnishes results intimately related to those obtained by employing the Hamiltonian as a Liapunov function, using variational coordinates for the generalized coordinates $q_1, \ldots, q_n$. This relationship simplifies stability analysis, permitting for this class of system the complete determination of conditions for stability or instability from *either* of the two following criteria.

Statement I: For Liapunov stability of a null solution of a system of nongyroscopic, holonomic, conservative differential equations for which $\partial L/\partial t = 0$, and for which the quadratic approximation of the dynamic potential energy P is neither positive nondefinite‡ nor zero, it is both necessary and sufficient that P be positive definite when expressed in terms of the variational coordinates.

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† The well-known dangers of omitting damping capacity from a mathematical model for purposes of stability analysis disappear for the particular class of dynamical systems treated here.

‡ A function $P(q_1, \ldots, q_n)$ is here called positive nondefinite if it is positive semidefinite but not positive definite or zero, i.e., if $P(0, \ldots, 0) = 0$ and in any neighborhood of the origin there are points for which $P = 0$ and points for which $P > 0$.

Statement II: For the determination of the Liapunov stability or instability of the null solution of a system of nongyroscopic, holonomic, conservative, dynamical equations for which $\partial L/\partial t$ is zero, analysis of the corresponding linearized variational matrix equation $A\ddot{q} + Kq = 0$ will suffice, unless the matrix K has a zero eigenvalue and no negative eigenvalues.

Interpretation

Equation (2), which must be satisfied for the system of equations given by Eq. (1) to qualify as nongyroscopic, can be written in an alternative form which often proves more convenient in application. If T_1 is substituted into Eq. (2) in the explicit form

$$T_1 = \sum_{j=1}^{n} \frac{\partial T_1}{\partial \dot{q}_j} \dot{q}_j \tag{8}$$

one finds

$$\frac{d}{dt}\left(\frac{\partial T_1}{\partial \dot{q}_i}\right) - \sum_{j=1}^{n} \frac{\partial^2 T_1}{\partial q_i \partial \dot{q}_j} \dot{q}_j = 0 \qquad i = 1, \ldots, n$$

or

$$\sum_{j=1}^{n} \frac{\partial^2 T_1}{\partial q_j \partial \dot{q}_i} \dot{q}_j + \frac{\partial^2 T_1}{\partial t \partial \dot{q}_i} - \sum_{j=1}^{n} \frac{\partial^2 T_1}{\partial q_i \partial \dot{q}_j} \dot{q}_j = 0 \tag{9}$$

The restriction $\partial L/\partial t = 0$ implies $\partial^2 T_1/\partial t \partial \dot{q}_i = 0$, so from Eq. (9) the equivalent of Eq. (2) is for the systems of interest

$$\sum_{j=1}^{n} \left(\frac{\partial^2 T_1}{\partial q_j \partial \dot{q}_i} - \frac{\partial^2 T_1}{\partial q_i \partial \dot{q}_j}\right) \dot{q}_j = 0 \quad i = 1, \ldots, n \tag{10}$$

which by virtue of the independence of the generalized coordinates requires

$$\frac{\partial^2 T_1}{\partial q_j \partial \dot{q}_i} = \frac{\partial^2 T_1}{\partial q_i \partial \dot{q}_j} \quad i,j = 1, \ldots, n \tag{11}$$

One may conclude from Eq. (11) that any system for which $n = 1$ is nongyroscopic, as is any system for which $T_1 = 0$, or for which $\partial T_1/\partial q_i = 0$, $i = 1, \ldots, n$.

The combination of Eqs. (1) and (2) furnishes the equations of motion

$$\frac{d}{dt}\left(\frac{\partial T_2}{\partial \dot{q}_i}\right) - \frac{\partial T_2}{\partial q_i} + \frac{\partial P}{\partial q_i} = 0 \quad i = 1, \ldots, n \tag{12}$$

The quadratic form T_2 may be written in matrix terms as

$$T_2 = \tfrac{1}{2}\dot{q}^T A \dot{q} \tag{13}$$

where A is a symmetric, positive definite n by n matrix, $\dot{q}$ is an n by 1 matrix of elements $\dot{q}_1, \ldots, \dot{q}_n$, and superscript T denotes transposition. A Taylor series expansion of the dynamic potential energy P about an assigned zero value at the origin of the variational coordinate space may be written in matrix terms as

$$P = \tfrac{1}{2}q^T K q + \cdots \tag{14}$$

where K is a symmetric n by n matrix, q is the n by 1 matrix with elements $q_1, \ldots, q_n$, and dots indicate terms of degree three and higher in $q_1, \ldots, q_n$. The absence from P of first-degree terms in $q_1, \ldots, q_n$ is a consequence of the assumption of the existence of the null solution of Eq. (1), while the presence of nonzero K is assured by the exclusion of functions P which are zero in quadratic approximation.

When Eqs. (14) and (13) are substituted into Eq. (12) and terms above the first degree in the variational coordinates $q_1, \ldots, q_n$ and their time derivatives are ignored, the result is the matrix equation

$$A\ddot{q} + Kq = 0 \tag{15}$$

The absence in Eq. (15) of a term proportional to $\dot{q}$ is characteristic of nongyroscopic holonomic conservative dynamical systems of linearized differential equations. [If there were to appear in Eq. (15) a term $G\dot{q}$, with G skew-symmetric, the equations would be gyroscopic, holonomic, and conservative, while the appearance of a term $D\dot{q}$ with D symmetric would indicate nongyroscopic, holonomic and nonconservative equations. For a dissipative, or damped, system, D is positive definite or positive semidefinite, and its influence is always stabilizing if $G = 0$, as in Eq. (15); hence the omission of damping in the model of a nongyroscopic system provides a safe approximation of a physical system.]

Proofs

Statement I and Statement II are most easily verified by demonstrating that for the system of equations under consideration the well-known sufficient conditions for Liapunov stability obtained by using the Hamiltonian as a Liapunov function must be satisfied when the sufficient conditions for instability obtained by well-established procedures from the linearized variational equations are violated, and vice versa.

The validity of Statement I is established by the following argument. The function P must be in one of the six mutually exclusive categories: positive definite, positive nondefinite, zero, negative nondefinite, negative definite, sign variable. If P is either positive nondefinite or zero, then P must be either positive nondefinite or zero in quadratic approximation, and the system of equations is excluded in Statement I. If P is either negative nondefinite, negative definite, or sign variable, then since P is neither positive nondefinite nor zero in quadratic approximation the matrix K must have a negative eigenvalue, and from Eq. (15) the null solution of Eq. (1) is unstable (given the positive definiteness of A). The only remaining possibility is that P be positive definite, so that by Eq. (6) H is positive definite (given the positive definiteness of T_2), and then since $\dot{H} = 0$ (see Eq. (7)) the Hamiltonian can be used as a Liapunov function to establish Liapunov stability of the null solution of Eq. (1). Thus for the class of equations in question it is both necessary and sufficient for Liapunov stability that P be positive definite, and Statement I is verified.

Confirmation of Statement II is accomplished similarly. The eigenvalues of the matrix K must be in one of three mutually exclusive categories: All positive, at least one negative, at least one zero and none negative. The last of these categories is excluded by Statement II. If any eigenvalue of K is negative, then from Eq. (15) the null solution of Eq. (1) is unstable (given the positive definiteness of A). In the only remaining alternative all eigenvalues of K are positive, so from Eq. (14) the function P is positive definite, leading by Eq. (6) to positive definiteness of H and thence by Liapunov's theorem to Liapunov stability of the null solution of Eq. (1), confirming Statement II.

Application

Of considerable practical interest in the field of aerospace vehicle attitude dynamics is the analysis of the motions of small bodies attached in various ways to vehicles of time-varying inertial orientation. If the motion of such a small body relative to a vehicle has no appreciable influence on the motion of the vehicle, then that relative motion may provide an indirect means of measuring the vehicle inertial attitude or rate of change of attitude; in other words, the small body may function as a sensor.

In the literature of satellite attitude dynamics, an early paper by Synge[8] and a recent paper by Kane and Athel[9] provide interesting examples of such analyses; in each of these papers the motion of the satellite is prescribed and the motion of a small pendulum within the satellite is examined. Synge's pendulum is a particle attached by a massless rod to the satellite mass center,[8] whereas the pendulum analyzed by Kane and Athel[9] is a rigid body C constrained to rotate about an axis fixed arbitrarily relative to a second rigid body B representing the primary body of the satellite (see Fig. 1).

Although the pendulum of Ref. 9 has but a single degree of freedom, its equation of motion is quite complex, requiring a full page of typed manuscript even for the special case in which

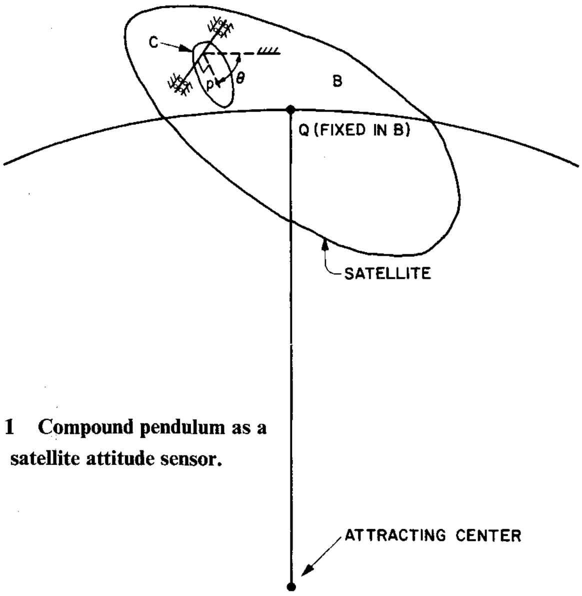

Fig. 1 Compound pendulum as a satellite attitude sensor.

body B is rotating at a constant rate with its inertial angular velocity normal to the orbital plane. When B maintains an Earth-pointing attitude, the angle θ which determines the rotation of the pendulum C relative to the body B must satisfy an equation of the form[9]

$$\ddot{\theta} + f_1 \sin\theta + f_2 \cos\theta + f_3 \sin 2\theta + f_4 \cos 2\theta = 0 \quad (16)$$

where $f_1, \ldots, f_4$ are constants available in terms of the system parameters. If θ^* is a constant solution of Eq. (16), and q is the variational coordinate

$$q \triangleq \theta - \theta^* \quad (17)$$

the second-order variational equation of motion becomes

$$\ddot{q} + f_1 \sin(\theta^* + q) + f_2 \cos(\theta^* + q) + f_3 \sin 2(\theta^* + q) + f_4 \cos 2(\theta^* + q) = 0 \quad (18)$$

Practical interest in this system stems from the fact that the solution θ^* depends on the attitude of a line fixed in B relative to the earth-pointing line through the mass center of B, so that θ^* is a measure of satellite attitude. In order for this device to function as an attitude sensor, the solution $\theta = \theta^*$ must of course be stable, so in Ref. 9 a stability analysis is provided.

Generally accepted formal procedures for stability analysis of undamped mechanical systems require firstly the determination of sufficient conditions for Liapunov stability by use of a Liapunov function (often the Hamiltonian), and secondly the determination of sufficient conditions for instability by means of the linearized variational equations. Kane and Athel proceed in this manner, discovering in this case that the Liapunov analysis provides the sufficient condition for stability

$$f_1 \cos\theta^* - f_2 \sin\theta^* + 2f_3 \cos 2\theta^* - 2f_4 \sin 2\theta^* > 0 \quad (19)$$

while linearization leads to the sufficient condition for instability

$$f_1 \cos\theta^* - f_2 \sin\theta^* + 2f_3 \cos 2\theta^* - 2f_4 \sin 2\theta^* < 0 \quad (20)$$

According to the criteria for stability and instability of the previous sections, it is possible on the basis of the Liapunov analysis alone to declare Eq. (19) both necessary and sufficient for stability (Statement I), or alternatively it is possible on the basis of linearized variational equations alone to declare Eq. (20) both necessary and sufficient for instability (Statement II); this simplification is a consequence of the fact that the system of Ref. 9 has but a single-degree-of-freedom ($n = 1$), so the equations are nongyroscopic (as well as holonomic and conservative). Application of the results of the present paper would therefore reduce the labors of stability analysis by more than half in the indicated problem of Ref. 9,§ and in many other problems of practical interest in mechanics.

Conclusions

Broadly speaking, one may say that the results of this paper have application to the following classes of problems: 1) single-degree-of-freedom holonomic, conservative, mechanical systems with $\partial L/\partial t = 0$; 2) holonomic, conservative, mechanical systems with $\partial L/\partial t = 0$ and with no rotating parts (so $T_1 = 0$); and 3) special multi-degree-of-freedom holonomic, conservative, mechanical systems which do have rotating parts but which also involve constraints which maintain $T_1 = 0$ (as is the case for example for the double compound pendulum on the constantly rotating base B illustrated in Fig. 2), or which maintain $\partial T_1/\partial q_i = 0$, $i = 1, \ldots, n$, or otherwise cause Eq. (2) to be satisfied. For the determination of stability criteria for solutions of the equations of motion of such mechanical systems, and more precisely for the determination of stability criteria for the null solution of nongyroscopic, holonomic, conservative, dynamical systems of differential equations with $\partial L/\partial t = 0$ and with certain mild restrictions on the dynamic potential energy P, it has been shown to be sufficient to analyze the corresponding linearized variational equations, or alternatively to examine the sign character of the Hamiltonian. (If one elects the latter option, then by retaining all terms in H one can, of course, obtain global stability criteria.) Since for the general case one must pursue both of these analytical procedures, and even then have no assurance that complete stability criteria will emerge, the simplifications described herein reduce the stability analysis task at least by half. The applicability of these results to problems of practical interest is illustrated by comparison with a recently published satellite attitude sensor analysis.

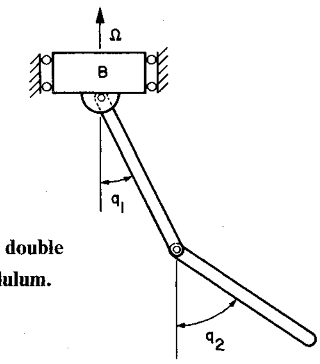

Fig. 2 Rotating double compound pendulum.

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§ The authors of Ref. 9 were aware of the theorems described herein, but they graciously deferred employing these theorems pending publication of this paper.